TUTORIAL

UNIT-WISE QUESTION BANK

ASSIGNMENT RECORD BOOK

DISCRETE MATHEMATICS

Course Code	**102903/MA300C**
Branch	**CS/IT/AD**
Semester	**III**
Academic Year	**2024 -2025**
Rajagiri School of Engineering and Technology (Autonomous)	

Name of the Student: ……………………………………………………….

Reg. No: …………………………Branch: …………………………..

Faculty in charge: ………………………………………………………..

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Name :	**Roll No:**
Branch:	

Module I - Fundamentals of Logic

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission:**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

Module II - Partial Order Relations and Lattices

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission:**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
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Module III - Generating Functions and Recurrence Relations

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission:**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
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Module IV - Algebraic Structures

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission:**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

Module V - Graph Theory

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission:**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

Total Marks:………………..

Signature of the faculty: ………………………

Bloom's Taxonomy with different difficulty levels

Unit wise Question Bank	Question Number	Difficulty Level
Module 1	1-5	C1, B3, C1, C2, C3
	6-10	C2, B3, D1, C3, C2
	11-15	C1, D1, C3, C2, B2
	16-20	B1, C1, B2, C2, D1
Module 2	1-5	B3, C2, A2, A1, A3
	6-10	D2, D3, C1, B3, C2
	11-15	C1, D1, C3, C2, B3
	16-20	A3, B1, B2, C2, C3
Module 3	1-5	B3, B1, B2, A3, A2
	6-10	C2, C1, B3, B2, C1
	11-15	B1, C2, B1, A1, A2
	16-20	B1, C2, A2, B2, B3
Module 4	1-5	A2, B3, D1, C3, C2
	6-10	A2, C3, C1, C2, D2
	11-15	C1, A3, B3, B2, C2
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Module 5	1-5	C1, A3, B3, B2, C2
	6-10	C2, A3, B1, D3, A2
	11-15	A2, A1, B1, C2, B3
	16-20	C2, B1, A2, D1, B2

			DIFFICULTY LEVEL		
			1	2	3
			LOW	MEDIUM	HIGH
Learning Objectives	A	Remember	A1	A2	A3
	B	Understand	B1	B2	B3
	C	Apply	C1	C2	C3
	D	Analyze	D1	D2	D3

MODULE 1

FUNDAMENTALS OF LOGIC

A **proposition** is a sentence which is either true or false. A proposition (denoted p, q, r,…) is simply a statement (i.e., a declarative sentence) with some definite meaning, (not vague or ambiguous) having a truth value that's either true (**T**) or false (**F**) .

The **negation** of p, written ∼p, is the statement obtained by negating statement p.

Truth values of p and ∼p are opposite. Symbol ~ is called "not" ~p is read as "not p

Example:

p: A is a consonant ~p: it is the case that A is not a consonant

r : "The Hudson is a big river." ¬r : "The Hudson is not a big river."

A **compound proposition** is built from propositions by the use of the **connectives** ``and", ``or" and ``not".

Truth Table of Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table of Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table of Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table of Bi-implication

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

.A **tautology** is a proposition which is always true. Eg: . p ∨ ¬p.

A **contradiction** is a proposition which is always false. p ∧ ¬p . Tautology is denoted by T_0 and Contradiction by F_0.

Two (compound) propositions s_1 and s_2 are said to be **equivalent** or logically equivalent, denoted by $s_1 \equiv s_2$, iff (i.e. if and only if) they have the same truth values. In other words for all possible truth values of the component statements, the compound propositions will have the same truth values.

- For any given proposition p →q, p is also known as the **premise** or **hypothesis** and q as the **conclusion**.
- **Converse** − The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If q, then p".
- ▶ q →p is the **converse** of p →q;
- **Inverse** − An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q".
- ▶ ~p →~q is the **inverse** of p →q;
- **Contra-positive** −.~q→~p

Let s be a statement. If s contains no logical connectives other than ∧ ,∨ then the dual of s denoted by s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

If p is the primitive statement, then p^d is the same as p – that is, the dual of a primitive statement is simply the same primitive statement. And $(\neg p)^d$ is the same as $\neg p$.

The statements $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other whenever p is primitive and so are the statements $p \vee T_0$ *and* $p \wedge F_0$.

Substitution rule

If we replace each occurrence of the primitive statement p by the compound statement $r \wedge s$ then the Tautology $(p \longrightarrow q) \leftrightarrow (\neg p \vee q)$ is again a tautology.

That is, $((r \wedge s) \longrightarrow q) \leftrightarrow (\neg(r \wedge s) \vee q)$ is a tautology.

1) Suppose that the compound statement P is a tautology. If p is a primitive statement that appears in P and we replace each occurrence of p by the same statement q, then the resulting compound statement P_1 is also a tautology.

2) Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Leftrightarrow p$. Suppose that in P we replace one or more occurences of p by q. Then this replacement yields the compound statement P_1 and $P_1 \Leftrightarrow P$.

Logical Implication: Rules of Inference

An **argument form**, or **argument** for short, is a sequence of statements. All statements but the last one are called **premises** or **hypotheses**. The final statement is called the **conclusion**, and is often preceded by a symbol ∴ . An argument is **valid** if the conclusion is true whenever all the premises are true. That is if the propositional form is a tautology. The validity of an argument can be tested through the use of the truth table.

Rules of Inference

Rules of inference are no more than valid arguments. The fundamental valid arguments are

- **modus ponens: modus tollens disjunctive addition: conjunctive addition: conjunctive simplification: disjunctive syllogism: hypothetical syllogism:division into cases: rule of contradiction:**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. In the sentence "The dog is sleeping": The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about. The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject. In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions. $P(x)$ = "x is sleeping" (where x is any object).The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable. For eg: $P(x, y) := x + 2 = y$, $x = 1$ and $y = 3$: $P(1,3)$ is true ; $x = 1$ and $y = 4$: $P(1,4)$ is false

Given a predicate P(x) and x has domain D, the truth set of P(x) is the set of all elements of D that make P(x) true. $\{x \in D, P(x)\}$

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol ∀. ∀x P(x) is read as for every value of x, P(x) is true.

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.

The variable x in each of open statements *p(x)* and *r(x)* is called a *free variable* (of the open statement). In contrast to the open statement *p(x)* the statement $\exists x$ *p(x)* has a fixed truth value — namely, true. And in the symbolic representation $\exists x$ *p(x)* the variable x is said to be a *bound* variable — it is bound by the existential quantifier ∃. This is also the case for the statements$\forall x$ *r* (*x*) and $\forall x \neg r(x)$, where in each case the variable x is bound by the universal quantifier ∀.

Let ***p(x) , q(x)*** be open statements defined for a given universe.

The open statements ***p(x)*** and ***q (x)*** are called (***logically***) ***equivalent***, and we write $\forall x[p(x) \Leftrightarrow q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement ***a*** from the universe (that is, $[p(x) \Leftrightarrow q(x)]$ for each ***a*** in the universe). If the implication $p(a) \longrightarrow q(a)$***)*** is true for each ***a*** in the universe (that is, $[p(a) \longrightarrow q(a)]$ for each ***a*** in the universe), then we write $\forall x[p(x) \Longrightarrow q(x)]$ and say that ***p (x) logically implies q(x).***

Rules for Negating Statements with One Quantifier

Negation of quantifiers: $\neg(\forall x \in D, P(x))$ is equivalent to $\exists x \in D, \neg P(x)$.

$\neg(\exists x \in D, P(x))$ is equivalent to $(\forall x \in D, \neg P(x))$.

Similarly, $\neg[(\forall x)\neg p(x)] \Leftrightarrow \exists x \neg\neg p(x) \Leftrightarrow \exists x p(x)$ and $\neg[\exists x \neg p(x)] \Leftrightarrow \forall x \neg\neg p(x) \Leftrightarrow \forall x p(x)$

TUTORIAL QUESTIONS

1. Determine whether each of the following sentences is a statement.

   a) In 2003 George W. Bush was the president of the United States.

   b) x + 3 is a positive integer.   c) Fifteen is an even number.

   d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

   e) What time is it? f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Let *p*, *q*, *r* denote the following statements about a particular triangle *ABC*.

   p: Triangle *ABC* is isosceles. *q:* Triangle *ABC* is equilateral. *r:* Triangle *ABC* is equiangular.

   Translate each of the following into an English sentence,

   a) $q \rightarrow p$ b) $\neg p \rightarrow \neg q$ c) $q \leftrightarrow r$ d) $p \wedge \neg q$ e) $r \rightarrow p$

3. Construct a truth table for each of the following compound statements, where *p* , *q* , *r* denote primitive statements. Which of the compound statements are tautologies? a) $\neg(p \vee \neg q) \rightarrow \neg p$ b) $p \rightarrow (q \rightarrow r)$ c) $(p \wedge q) \rightarrow p$

4. Let *p* , *q* , *r* denote primitive statements.

   a) Use truth tables to verify the following logical equivalences.

   i) $[p \rightarrow (q \wedge r)] \Leftrightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$ ii) $[p \rightarrow (q \vee r)] \Leftrightarrow [\neg r \rightarrow (p \rightarrow q)]$

   b) Use the substitution rules to show that $[[[\boldsymbol{p} \rightarrow (\boldsymbol{q} \vee \boldsymbol{r})] \Leftrightarrow [(p \wedge \neg q) \rightarrow r]$

5. For primitive statements *p* , *q* , *r* , and *s*, simplify the compound statement

   $[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \rightarrow s$

ASSIGNMENT QUESTIONS

1. Using the statements   R: Mark is rich.   H: Mark is happy.

   Write the following statements in symbolic form: Mark is poor but happy.

   Mark is rich or unhappy. Mark is neither rich nor unhappy. Mark is poor or he is both rich & unhappy.

2. Assuming that *p* & *q* are F & *q* & *s* are T, find the truth value of each proposition:

   $p \rightarrow q$ $\neg p \rightarrow \neg q$ $\neg(p \rightarrow q)$ $(p \rightarrow q) \wedge (q \rightarrow r)$ $(p \rightarrow q) \rightarrow r$ $p \rightarrow (q \rightarrow r)$

   $(s \rightarrow (p \wedge \neg r)) \wedge ((p \rightarrow (r \vee q)) \wedge s)$ $((p \wedge \neg q) \rightarrow (q \wedge r)) \rightarrow (s \vee \neg q)$

3. Rewrite the following statements without using the conditional:

If it is cold, he wears a hat.   If productivity increases, then wages rise.

4. Determine the converse, inverse and contrapositive of each statement:

If John is a poet, then he is poor. Only if Marc studies will he pass the test.

5. Write the negation of each statement as simply as possible:

If she works, she will earn money. He swims if and only if the water is warm.

If it snows, then they do not drive the car.

6. Formulate the symbolic expressions in words using:

p: Today is morning, q: It is raining, r: It is hot.

$p \rightarrow q$ $\neg q \rightarrow (r \wedge p)$ $\neg p \rightarrow (q \vee r)$ $\neg(p \vee q) \leftrightarrow r$ $(p \wedge (q \vee r)) \rightarrow (r \vee (q \vee p))$

7. Show that: $(p \rightarrow q) \equiv (\neg p \vee q)$, $(p \leftrightarrow q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

8. a) Test the validity of each argument: If it rains, Erik will be sick. It did not rain.

$\therefore$ Erik was not sick.

b) If it study, then I will not fail mathematics. If I do not play basketball, then I will study. But I failed mathematics. Therefore I must have played basketball.

9. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ & $\neg M$.

10. Show that $\neg(p \wedge q)$ *follows from* $\neg p \wedge \neg q$.

UNIT WISE QUESTION BANK

1. Determine the truth value of each of the following:

a)   4+2 =6 and 2+2 =4 . b) 5+4 =9 and 3+3 = 5. c) 6+4 = 10and 1+1= 3.

2. Find the converse, inverse, and contra positive of the following statement

a) p; Charminar is in Hyderabad q; 7+1= 6.   b) .p; It is not true that Delhi is in Russia. q: Russia is not in Europe.

3. Construct truth tables for the following:

i) $\neg (p \vee \neg q) \rightarrow q$. ii). $(p \wedge q) \vee (p \wedge q) \rightarrow q$. iii). $p \wedge (q \wedge p)$.

4. i).Show that $(p\wedge q)\rightarrow p$ is a tautology. ii)Show that $\neg(p\rightarrow q) \equiv p\wedge q$.

5. Show that $(p\wedge(\neg p\vee q))\vee(q \wedge\neg(p\wedge q))\equiv q$

6. Write the duals of i) $(p\vee q)\wedge r$ ii) $(p\wedge q)\vee t$

7. Write the dual of $\neg(p\vee q) \wedge(p\vee\neg(q\wedge\neg s))$.

8. Define the following terms: i)Tautology ii) Fundamental connectives

9. Consider the following statements: P : He is coward. Q: He is rich. R: He is lazy

Write the following compound statements in the symbolic form.

He is neither coward nor lazy. It is false that he is coward but not lazy. He is rich or else he is both coward and lazy. He is coward or lazy but not rich.

10. Let *p, q, r* denote the following statements: *p*: Triangle ABC is isosceles. *q:* Triangle ABC is equilateral. *r*: Triangle ABC is equiangular. Translate each of the following into a statement of English. (a) $p\rightarrow q$ (b) $\neg p\rightarrow\neg q$ (c) $q\leftrightarrow r$ (d) $r\rightarrow p$

11. Show that $((P\rightarrow Q)\rightarrow R)\Leftrightarrow(P\rightarrow R)\vee(Q\rightarrow R)$

12. Show that $((p\vee q)\wedge\neg(\neg p\wedge(\neg q\vee\neg r)))\vee(\neg p\wedge\neg q)\vee(\neg p\wedge\neg r)$ is a tautology

13. Show that $S\vee R$ is tautologically implied by $(P\vee Q)\wedge(P\rightarrow R)\wedge(Q\rightarrow S)$

14. Using truth table show that $(A\rightarrow(B\rightarrow C))\Rightarrow(A\rightarrow B)\rightarrow(A\rightarrow C)$

15. Using duality law, show that $(p\vee q)\wedge(\neg p\wedge(\neg p\wedge q))\Leftrightarrow\neg p\wedge q$

16. Show that $\neg(p\rightarrow q) \equiv p\wedge\neg q$.

17. Let T(x, y) be the propositional function "x is taller than y." The domain of discourse consists of three students: Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Mary who is 6 feet tall. Write the following propositions in word & tell whether it is T or F. Also write negation of each proposition in words & symbolically.

a) $\forall x \forall y\, T(x, y)$ b). $\forall x \exists y\, T(x, y)$ c) $\exists x \forall y\, T(x, y)$ d). $\exists x \exists y\, T(x, y)$

18. Negate each of the following and simplify the resulting statement.

a) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$ b) $(p \wedge q) \rightarrow r$

19. a) Write the dual for (a) $q \to p$, (b) $p \to (q \wedge r)$ (c) $p \leftrightarrow q$where p , q , and r are primitive statements.

b) Write the converse, inverse, and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth values of its corresponding converse, inverse, and contrapositive.

i) If 0 + 0 = 0, then 1 + 1 = 1.   ii) If -1 < 3 and 3 + 7 = 10, then $in\ (\frac{3\pi}{2}) = -1$.

20. a) Consider each of the following arguments. If the argument is valid, identify the rule of inference that establishes its validity. If not, indicate whether the error is due to an attempt to argue by the converse or by the inverse.

i) Andrea can program in C++, and she can program in Java. Therefore Andrea can program in C++.

ii) If Ron's computer program is correct, then he'll be able to complete his computer science assignment in at most two hours.It takes Ron over two hours to complete his computer science assignment.

Therefore Ron's computer program is not correct.

iii) Eileen's car keys are in her purse, or they are on the kitchen table. Eileen's car keys are not on the kitchen table.

Therefore Eileen's car keys are in her purse.

v) If interest rates fall, then the stock market will rise.Interest rates are not falling.

Therefore the stock market will not rise.

b) Let *p(x)*, *q(x)*, and r(x) denote the following open statements.

$px): x^2 - 8x + 15 = 0, q(x): x\ is\ odd, r(x): x > 0$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

i) $\forall x[p(x) \to q(x)]$ $ii)$ $\forall x[p(x) \to q(x)]$ $iii)$ $\exists x[p(x) \to q(x)]$ $iv)$ $\exists x[r(x) \to p(x)]$

$v)$ $\exists x[p(x) \to (q(x) \wedge r(x)]$

MODULE 2

Partial Order Relations and Lattices

Cartesian product

For sets A, B the ***Cartesian product*, or *cross product***, of A and B is denoted by $A \times B$ and equals $\{(a, b) / a \in A, b \in B\}$

Relation or Binary Relation:

For sets A, B, any subset of $A \times B$ is called a **(*binary) relation*** from A to B. Any subset of $A \times A$ is called a (*binary) relation* on A.

Note:

For finite sets A, B with $|A| = m$ and $|B| = n$, there are 2^{mn} relations from *A* to *B*, including the empty relation as well as the relation $A \times B$ itself.

Function – domain, range-one to one function

For nonempty sets A, B, a **function or mapping** f from A to B, denoted $f: A \rightarrow B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

A function $f: A \rightarrow B$ is called **one-to-one or injective**, if each element of B appears at most once as the image of an element of A.

If $f: A \rightarrow B$ and $A_i \subseteq A$ then, $f(A_i) = \{b \in B / b = f(a), a \in A_i\}$ and $f(A_i)$ is called the **image** of A_i under f.

If $f: A \rightarrow B$, with $A_1 \subseteq A$, then $f /_{A_1}: A_1 \rightarrow B$ is called **restriction** of f to A_1 if $f /_{A_1}(a) = f(a)$ for all $a \in A_1$.

Properties of Relations:

A Relation R on a set A is called **reflexive** if for all $x \in A, (x, x) \in R$.

i.e. $xRx\ \forall x \in A$.

Given a finite set A with $|A| = n$, then there are $2^{(n^2-n)}$ reflexive relations on A.

A Relation R on a set A is called **symmetric** if for $(x, y) \in R \Rightarrow (y, x) \in R$,for all $x, y \in A$.

i.e. if xRy then yRx.

A Relation R on a set A is called **transitive** if for all $x, y, z \in A, (x, y), (y, z) \in R \Rightarrow (x, z) \in R$.

i.e. if x "is related to" y and y "is related to" z then x "is related to" z.

i.e. if xRy and yRz then xRz .

A Relation R on a set A is called **antisymmetric** if for all $a, b \in A, aRb$ and $bRa \Rightarrow a = b$.

A relation R on a set A is called **partial order** or a partial ordering relation, if R is reflexive, antisymmetric and transitive.

A relation R on a set A is called **equivalence relation**, if R is reflexive, symmetric and transitive.

A Relation R on a set A is called **irreflexive** if for all $x \in A, (x, x) \notin R$.

i.e. $xRx\ \forall x \in \mathbb{Z}$.

Partially ordered Set – Hasse Diagram:-

A non-empty set S together with a partial ordering R is called a **Partial ordered set or Poset** iff R is reflexive, antisymmetric and transitive. Partial ordering is denoted by $' \leq '$ and $'a \leq b'$ is read as $'a'$ preceeds $'b'$. Then $(S, \leq)$ is called a Partial ordered set or Poset.

A partial ordering $\leq$ on a set P can be represented by means of a diagram known as **Hasse diagram** of the poset $(p, \leq)$. In such a diagram, each element is represented by a circle or a dot.

Maximal & Minimal Elements:-

If $(A, \mathbb{R})$ is a poset, then an element $x \in A$ is called a **maximal** element of A if for all $a \in A, a \neq x \Longrightarrow x$ not related to a. An element $y \in A$ is called a **minimal** element of A if whenever $b \in A, b \neq y \Longrightarrow b$ not related to y.

Upper Bound & Lower Bound:-

Let (A, R) be a poset with $B \subseteq A$. An element $x \in A$ is called a **lower bound** of B if xRb for all $b \in B$. Likewise, an element $y \in A$ is called an **upper bound** of B if bRy for all $b \in B$.

An element $x' \in A$ is called a **greatest lower bound** (glb) of B if it is a lower bound of B and if for all other lower bounds x'' of B we have $x''Rx'$. Similarly $y' \in A$ is a least upper bound (lub) of B if it is an upper bound of B and if $y'Ry''$ for all other upper bounds y'' of B.

Equivalence Relations and Partitions - Equivalence Class

Suppose R is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of S to which a is related under R,

$$\text{i.e. } [a] = \{x : (a, x) \in R\}$$

we call $[a]$, the **equivalence class** of a in S.

Given a set A and index set I, let $\emptyset \neq A_i \subseteq A$ for each $i \in I$. Then $\{A_i\}_{i \in I}$ is a **partition** of A if

a) $A = \bigcup_{i \in I} A_i$ and **b)** $A_i \cap A_j = \emptyset$, for all $i, j \in I$ where $i \neq j$.

Each subset A_i is called a cell or block of the partition.

Lattice :-

A partially ordered set $(A, \leq)$ in which every pair of elements has a least upper bound and greatest lower bound is called a **lattice** .

Dual or reversed lattice :-

If $(A, \leq) = [A, \vee, \wedge]$ is a lattice then $(A, \geq) = [A, \wedge, \vee]$ is known as dual or reversed lattice, which is obtained by interchanging *glb and lub*.

Sublattice : Let (A, R) be a lattice. A non-empty subset B of A is called a sublattice if for any $a, b \in B, a \vee b \text{ and } a \wedge b \in B$.

Properties of glb and lub:

1. $x * y \leq x$ and $x * y \leq y$ 2. $m \leq x$ and $m \leq y \Rightarrow m \leq x * y$

3. $x \leq x + y$ and $y \leq x + y$ 4. $x \leq u$ and $y \leq u \Rightarrow x + y \leq u$

Complete Lattice*:* A lattice is called complete if each of its nonempty subsets has a least upper bound and a greatest lower bound.

Bounded lattice: A lattice is said to be bounded if it has a greatest element 1 and least element 0.

Complement: In a bounded lattice, an element $b \in L$ is called a complement of an element $a \in L$, if $a * b = O$ and $a + b = I$.

Complemented lattice:A lattice is said to be a complemented lattice if every element of L has at least one complement.

Distributive lattice: A lattice $[A,*,+]$ is called a distributive lattice if for any $a, bc \in A$,

$a * (b + c) = (a * b) + (a * c)$ and $a + (b * c) = (a + b) * (a + c)$

Tutorial Questions

1. Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
2. Let $A = \{1,2,3,4\}$ and $B = \{x, y, z\}$.(a) List five functions from A to B. (b) How many functions $f: A \longrightarrow B$ are there? (c) How many functions $f: A \longrightarrow B$ are one to one?
3. Let A be a set of non-integers and $\approx$ be a relation on$(a, b) \approx (c, d)$ whenever $ad = bc$. Prove that $\approx$ is an equivalence relation.
4. Let $A = \{1,2,3,6,9,18\}$ and define $\mathcal{R}$ on A by $x\mathcal{R}y$ if x divides y. Draw the Hasse diagram for the poset $(\mathcal{R},/)$.
5. Draw Hasse diagram and determine which of the following posets $(A,/)$ are lattices and why?
   a) $A = \{1,2,3,5,30\}$ b) $A = \{1,2,3,4,6\}$ c) $A = \{1,2,3,4,12\}$ d) $A = \{2,3,4,12\}$
   e) $A = \{1,3,6,30\}$ f) $A = \{1,2,3,4,6,12\}$

Assignment Questions

1. If $A = \{1,2,3,4\}$, $B = \{2,5\}$ and $C = \{3,4,7\}$. Determine
   (a) $A \times B$ (b) $B \times A$ (c) $A \cup (B \times C)$ (d) $(A \cup B) \times C$ (e) $(A \times C) \cup (B \times C)$.
2. Let $A = \{1,2,3\}$ and , $B = \{1,2,3,4,5\}$. Check whether the following functions are one to one.
   (i) $f = \{(1,1), (2,3), (3,4)\}$ (ii) $g = \{(1,1), (2,3), (3,3)\}$
3. Determine whether or not each of the following relations are a function.

(a) $\{(x, y): x, y \in \mathbb{Z}, y = x^2 + 7\}$ is a relation from $\mathbb{Z}$ to $\mathbb{Z}$.

(b) $\{(x, y): x, y \in \mathbb{R}, y = 3x + 1\}$ is a relation from $\mathbb{R}$ to $\mathbb{R}$.

4. If $A = \{1,2,3,4,5\}$ give an example of a relation R on A that is

   (a) Reflexive and Symmetric but not Transitive

   (b) Reflexive and Transitive but not Symmetric

   (c) Symmetric and Transitive but not Reflexive.

5. Draw the Hasse diagram for the poset $(\mathcal{P}(\mathcal{U}), \subseteq)$, where $\mathcal{U} = \{1,2,3,4\}$.

6. If $A = \{1,2,3,4,5,6,7\}$, define $\mathcal{R}$ on A by $(x, y) \in \mathcal{R}$ if $x - y$ is a multiple of 3.

   (a) Show that $\mathcal{R}$ is an equivalence relation on A.

   (b) Determine the equivalence classes and partition of A induced by $\mathcal{R}$.

7. Is the poset $(A, \ /\ )$ where $A = \{2,4,6,24,36,72\}$ a lattice? Explain. Draw Hasse diagram.

8. Let $A = \{1,2\}, B = \{1,3\}, C = (1,2,3\}$ and $X = \{A, B, C\}$. a) Draw Hasse diagram for $(X, \subseteq)$ b) Is $(X, \subseteq)$ a lattice? C) Does $glb\ (A, B) = A \cap B$?

9. Show that the lattice $(A, \ /)$, where $\{1,2,3,4,12\}$, $*\ and +$ are not distributive.

10. Let P = {1, 2, 3, 5, 6, 10, 15, 30} with R = D then (P, D) is a poset and let A = {6, 10}. Then find the lower bound of A.

<u>Unitwise Question bank</u>

1. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

2. Which of the following are one to one functions:

   a) $g(x) = x + 7 \quad b)\ g(x) = 2x - 3$

3. Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, /) are maximal and which are minimal?

4. Prove that $(\boldsymbol{P}(A), \subseteq)$ is a poset? We have to show that $\subseteq$ is reflexive, antisymmetric, and transitive.

5. For each of the following functions, determine whether it is one to one:

   (a) $f: \mathbb{Z} \longrightarrow \mathbb{Z}, f(x) = 2x + 1 \quad b): \mathbb{R} \longrightarrow \mathbb{R}, f(x) = e^x$

6. Determine the number of relations on $A = \{a, b, c, d, e\}$ that are (a) Reflexive (b) Symmetric (c) Reflexive and Symmetric (d) Antisymmetric.

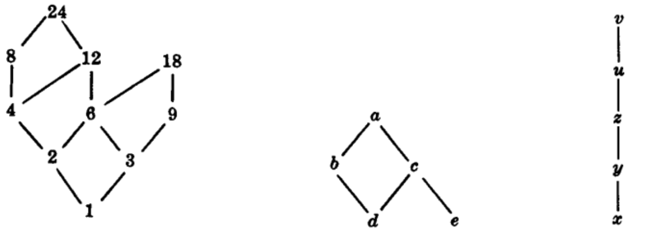
7. Give an example of a relation $\mathcal{R}$ on Z where $\mathcal{R}$ is irreflexive and transitive but not symmetric.

8. If | A | = n > 1, how many different relations on A are irreflexive? How many are neither reflexive nor irreflexive?

9. Let (A, $\mathcal{R}_{1}$), (B, $\mathcal{R}_2$) be two posets. On A X B, define relation $\mathcal{R}$ by (a, b) $\mathcal{R}$(x, y) if a$\mathcal{R}_1$x and b$\mathcal{R}_2$y . Prove that $\mathcal{R}$ is a partial order.

10. Let A = {1, 2, 3, 4, 5, 6, 7}. How many symmetric relations on A contain exactly (a) four ordered pairs? (b) five ordered pairs? (c) seven ordered pairs? (d) eight ordered pairs?

11. Consider



(a) (b) (c)

Find the Maximal element, Minimal Element, Greatest element and least element.

12. Define the relation R on the set of positive integers by (x, y) ∈ R if the greatest common divisor of x and y is 1. Determine whether R is reflexive, symmetric, antisymmetric, transitive, and/ or a partial order.
13. If A = {1, 2, 3, 4, 5} and $\mathcal{R}$ is the equivalence relation on A that induces the partition A = {1, 2} U {3, 4} U {5}, what is $\mathcal{R}$?
14. If | A | = 30 and the equivalence relation $\mathcal{R}$ on A partitions A into (disjoint) equivalence classes A_1, A_2, and A_3, where $|A_1| = |A_2| = |A_3|$, what is | $\mathcal{R}$ |.
15. Let D_m be the set of integers of a particular positive integer $'m'$ and let ≤ be the relation divides. Draw the Hasse diagram for $m = 210$.
16. Let A = {1, 2, 3, 4, 5} X {1, 2, 3, 4, 5}, and define $\mathcal{R}$ on A by (x_1, y_1) $\mathcal{R}$ (x_2, y_2) if $x_1 + y_1 = x_2 + y_2$. Verify that $\mathcal{R}$ is an equivalence relation on A.
17. If A = {1, 2, 3, 4, 5, 6, 7}, define $\mathcal{R}$ on A by (x, y)∈ $\mathcal{R}$ if x - y is a multiple of 3. Show that $\mathcal{R}$ is an equivalence relation on A.
18. For A = {(-4 , -20), (-3, -9), (-2 , -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12)} define the relation $\mathcal{R}$ on A by (a, b) $\mathcal{R}$ (c, d) if ad = bc. Find the equivalence classes [(2, 14)], [(-3, -9)], and [(4, 8)] .
19. Let f : A→B. If {B_1, B_2, B_3, …… B_n} is a partition of B, prove that $\{f^{-1}(B_1) | 1 \leq i \leq n, f^{-1}(B_1) \neq \emptyset\}$ is a partition of A.

20.Find the complements for the following lattice (A, R) where

(a) $A = \{1,2,3,6\}$ (b) $A = \{1,2,3,5,30\}$

Module- 3

Generating Functions and Recurrence Relations

First order linear recurrence relations with constant Coefficients:

The general form of first order linear homogeneous recurrence relation with constant coefficients is given by, $a_{n+1} = d\, a_n, n \geq 0$, where d is a constant. Values such as a_0 or $a_1$are called boundary condition. The expression $a_0 = A$, where A is a constant is called the initial condition.The unique solution of the recurrence relation $a_{n+1} = d\, a_n, n \geq 0$, where d is a constant and $a_0 = A$, is given by $a_n = Ad^n, n \geq 0$.

Second order Linear recurrence relation with constant coefficients:

A linear recurrence relation with constant coefficients is of the form
$c_o a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$ where $c_i's$ are constants. The order of the relation is k.

When$f(n) = 0$, then the equation is a homogeneous linear recurrence relation.

When $f(n) \neq 0$, then the equation is a non-homogeneous linear recurrence relation.

Consider the homogeneous linear recurrence relation of the form $c_o a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0, n \geq 2$. Substituting $a_n = cr^n$ into $c_o a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$, we obtain

$$c_0 cr^n + c_1 cr^{n-1} + c_2 cr^{n-2} = 0$$

With $c, r \neq 0$, this becomes $c_0 r^2 + c_1 r + c_2 = 0$ a quadratic equation called the characteristic equation. The roots $r_{1,} r_2$ determine the following three cases: (a) $r_{1,} r_2$ are real and distinct (b) $r_{1,} r_2$ are complex conjugate (a) $r_{1,} r_2$ are real and equal. In all cases $r_{1,} r_2$ are called the characteristic roots.

Characteristic Roots	Solution
$r_{1,} r_2, \ldots\ldots, r_k$ distinct roots	$a_n = c_1 {r_1}^n + c_2 {r_2}^n + \cdots + c_k {r_k}^n$,where $c_1, c_2, \ldots, c_k$ are constants
$r_{1,} r_2$ are repeated roots	$a_n = (c_1 + c_2 n) r^n + c_3 {r_3}^n + \cdots + c_k {r_k}^n$

Non-homogeneous Recurrence Relation:

A non-homogeneous linear recurrence relation with constant coefficients is of the form $c_o a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$ where $c_i's$ are constants and $f(n) \neq 0$. The order of the relation is k. The solution of the above equation is given by,

$a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(h)}$ is the solution of the associated linear homogeneous recurrence relation and $a_n^{(p)}$ is the particular equation.

	f(n)	Trial Solution
1.	b^n (If b is not a root of characteristic equation)	Ab^n
2.	b^n (If b is a root of characteristic equation of multiplicity l)	$An^l b^n$
3.	$p(n)$, a polynomial of degree m.	$A_0 + A_1 n + \cdots + A_m n^m$
4.	$c^n\, p(n)$, (c is a constant and c is not a root of the characteristic equation)	$c^n(A_0 + A_1 n + \cdots + A_m n^m)$
5.	$c^n\, p(n)$, (c is a constant and c is a root of multiplicity l)	$n^l c^n(A_0 + A_1 n + \ldots + A_m n^m)$

Generating Function - Definition and Examples - Calculation techniques:

Definition: Let a_0, a_1, a_2 ... be a sequence of real numbers, the function $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$ is called the generating function for the given sequence. Where a_n is the coefficient of x^nin the generating function is called the discreate numeric function.

Example: The generating function for the sequence 1,1,1, ... is given by $f(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$.

TUTORIAL QUESTIONS

1. Find the generating function for the following sequence
   (a) $1, -1, 1, -1, 1, -1, \ldots$
   (b) $0, 0, 1, a, a^2, a^3, \ldots, a \neq 0$
   (c) $0, 0, 0, 6, -6, 6, -6, \ldots$
2. Determine the sequence generated by each of the following generating functions
   (a) $f(x) = (2x - 3)^3$ $\quad b)\ \frac{x^4}{(1-x)}$ $\quad c)\ \frac{1}{(3-x)}$
3. Find the first four terms of the following recurrence relation, $a_k = 2a_{k-1} + k, \forall k \geq 2, a_1 = 1$
4. Solve the following recurrence relation
   (a) $a_{n+2} + a_n = 0. n \geq 0, a_0 = 0, a_1 = 3.$
   (b) $a_n - 6a_{n-1} + 9a_{n-2} = 0, n \geq 2, a_0 = 5, a_1 = 12.$
5. Solve the following recurrence relations
   (a) $a_{n+1} - a_n = 2n + 3, n \geq 0, a_0 = 1$

(b) $a_{n+1} - 2a_n = 2^n, n \geq 0, a_0 = 1$

ASSIGNMENT QUESTIONS

1. Find the first four terms of the following recurrence relation

   (a) $a_n = a_{n-1} + 3a_{n-2}, \forall n \geq 3, a_1 = 1, a_2 = 2.$

   (b) $a_n = n(a_{n-1})^2 \forall n \geq 2, a_1 = 1.$

2. Solve the following recurrence relations.

   (a) $a_n = 5a_{n-1} + 6a_{n-2}, \forall n \geq 3, a_0 = 1, a_1 = 3.$

   (b) $2a_{n+2} - 11a_{n+1} + 5a_n = 0, \forall n \geq 0, a_0 = 2, a_1 = -8$

3. Solve the homogeneous recurrence relation $a_n + 4a_{n-1} + 2a_{n-2} = 0, \forall n \geq 2, a_0 = 1, a_1 = 3.$

4. If $a_0 = 0, a_1 = 1, a_2 = 4, a_3 = 37$ satisfying the recurrence relation $a_{n+2} + ba_{n+1} + ca_n = 0,$ where $n \geq 0$ and b, c are constants, determine b, c and solve for a_n.

5. The population of Mumbai city is 6,000,000 at the end of the year 2000. The number of immigrants is $20{,}000n$ at the end of the year n. The population of the city increases at the rate of 5% per year. Use a recurrence relation to determine the population of the city at the end of 2010.

6. Find a recurrence relation, with initial condition, that uniquely determines each of the following geometric progressions.

   (a) $2, 10, 50, 250, \ldots$

   (b) $6, -18, 54, -162, \ldots$

7. Find the solution for the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = g(n)$ where (a) $g(n) = 3^n$ (b) $g(n) = n3^n$ (c) $g(n) = (1 + n^2)3^n$

8. Solve the recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = 0$ by the method of generating function with the initial condition $a_0 = 2, a_1 = 3.$

9. Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$ by the method of generating function with the initial condition $a_0 = 2, a_1 = 1.$

10. Find the general solution of the non-homogeneous recurrence relation $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 3 + 5n, n \geq 0.$

UNIT WISE QUESTION BANK

1. Find its generating function for the sequence $\{ 3^0, 3^1, \ldots 3^r, \ldots.\}$

2. Solve the relation $a_n + a_{n-1} - 6a_{n-2} = 0$ with the initial conditions $a_0 = -1, a_1 = 8$

3. Solve the relation $a_n + 10a_{n-1} - 25a_{n-2} = 0$ with the initial conditions $a_0 = 0, a_1 = 4$
4. Solve the relation $5a_n - 11a_{n-1} + 2a_{n-2} = 0$ with initial conditions, $a_0 = 2, a_1 = -8$
5. Solve the relation $4a_n + 4a_{n+1} + a_{n+2} = 0$ with initial conditions, $a_0 = 0, a_1 = 1$
6. Solve the relation $a_n - 2a_{n-1} - 6a_{n-2} = 0$ with initial conditions, $a_0 = -1, a_1 = 8$
7. Find the general solution of the recurrence relation by the method of characteristic root $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$.
8. Find the general solution of the recurrence relation by the method of characteristic root (a) $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$. b) $a_n = 3a_{n-1} - 4n + 3(2^n)$
9. Apply the generating function method to solve the following recurrence relation $a_{n+2} + 4a_{n+1} + 4a_n = 0; \quad a_0 = 1, a_1 = 0$
10. Find a closed form of the generating function for each of the following sequence. (a) $0,0,1,1,1, \ldots$ $b)$ $1,1,0,1,1,1,1,1, \ldots$
11. Show that $a_n = -2^{n+1}$ is a solution of the non-homogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.
12. Show that $a_n = \left(\frac{1}{6}\right) 5^n$ is a solution of the non-homogeneous linear recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$.
13. Apply the generating function method to solve the following recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = 0; \quad a_0 = 2, a_1 = 3$
14. Use the iterative method to find the solution to the following recurrence relation $a_n = na_{n-1}, a_0 = 5$.
15. Solve the recurrence relation : $a_{n+2} + 2a_{n+1} - 15a_n = 6n + 10; \quad a_0 = 1, a_1 = -\frac{1}{2}$.
16. Apply generating function technique to solve the recurrence relation $a_n = 4(a_{n-1} - a_{n-2}) + 2^n$
17. Apply generating function technique to solve the recurrence relation $a_{n+2} + 2a_{n+1} + a_n = 1 + n$
18. Find the sequence $\{a_n\}$ having the generating function $G(x)$ given by (a) $\frac{x}{(1-2x)}$ (b) $\frac{x}{(1-x)^2} + \frac{x}{(1-x)}$.
19. Solve the recurrence relation by the generating function method $a_n - 3a_{n-1} = n, n \geq 1, a_0 = 1$.
20. Find the generating function for the discrete numeric function $a_n = 2^n + 3, n \geq 0$.

Module 4

Algebraic Structures

Algebraic system:

An algebraic system or simply an algebra is a system consisting of a non empty set A and one or more n-ary operations on the set A. It is denoted by $< A, f_1, f_2, \ldots >$ where $f_1, f_2, \ldots$ are operations on A.

An algebraic structure is an algebraic system $< A, f_1, f_2, \ldots, R_1, R_2, \ldots >$ wherein addition to operations f_i ,the relation R_i are defined on A. this leads to structure on elements of A.

General Properties for an algebraic structure:

Let A be a non empty set and $+$ and **.** (not necessarily the usual addition and multiplication) be any two binary operations on A. Then for any elements a, b, c of A we have

(1) Associative Property for $+$:

$$(a + b) + c = a + (b + c)$$

(2) Commutative Property for $+$:

$$a + b = b + a$$

(3) Identity Element 0 for + :

There exists an identity element $0 \in A$ such that for any $a \in A$

$$a + 0 = 0 + a = a$$

(4) Inverse element under $+$:

For each $a \in A$, there exists an element $b \in A$(called the negative of a) such that

$$a + b = b + a = 0$$

(5) Associative Property for **.** :

$$(a. b). c = a. (b. c)$$

(6) Commutative Property for **.** :

$$a. b = b. a$$

(7) Identity Element 1 for. :

$$a. 1 = 1. a = a$$

(8) Distributive law of **.** over $+$:

a) $a. (b + c) = (a. b) + (a. c)$

b) $(b + c)$.a = (b.a)+(c.a)

(9) Cancellation Property:

$$a.b = a.c \Longrightarrow b = c \text{ , provided } a \neq 0$$

(10) Idempotent Property:

$$a + a = a; \qquad a.a = a$$

Homomorphism and Isomorphism:

Let $(X,.)$ and $(Y,*)$ be two algebraic systems where $.,*$ are both n-ary operations. A function $f: X \to Y$ is known as homomorphism from $(X,.)$ to $(Y,*)$ if for any $x_1, x_2 \in X$ we have

$$f(x_1.x_2) = f(x_1) * f(x_2)$$

Thus the operations are preserved in a homeomorphism. A homeomorphism is also known as epimorphism, monomorphism or isomorphism if f is onto, one to one or one to one onto respectively.

Semigroups and Monoids:

An algebraic structure $(S,.)$ is known as semigroup where S is a nonempty set, and . is a binary operation which is associative. Further if . is commutative then the semigroup is said to be commutative(or abelian) semigroup.

A monoid $(M,.)$ is a semigroup with an identity e . Monoid may be represented as $(M,.,e)$.

Subsemigroups and Submonoids:

Let $(S,*)$ be a semigroup and $T \subseteq S$. Then $(T,*)$ is said to be subsemigroup of $(S,*)$ if T is closed under the operation $*$.

Let $(M,*,e)$ be a monoid and $T \subseteq M$.Then $(T,*,e)$ is known as a submonoid if T is closed under the operation $*$ and the identity $e \in T$.

Group:

If G is a nonempty set and o is a binary operation on G, then (G, o) is called a group if the following conditions are satisfied.

1) For all $a, b \in G, a\ o\ b \in G$. (Closure of G under o)

2) For all $a, b, c \in G, a\ o\ (b\ o\ c) = (a\ o\ b)\ o\ c$. (The Associative Property)

3) There exists $\in G\ with\ a\ o\ e = e\ o\ a = a$, for all a ∈ G. (The Existence of an Identity)

4) For each $a \in G$ there is an element $b \in G$ such that $a\ o\ b = b\ o\ a = e$. (Existence of Inverses)

Furthermore, if $a\ o\ b = b\ o\ a$ for all a , $b \in G$, then G is called a *commutative*, or *abelian* group.

Definition:

For every group G the number of elements in G is called the *order* of G and this is denoted by $|G|$. When the number of elements in a group is not finite we say that G has infinite order.

Theorem

For every group G,

a) the identity of G is unique.

b) the inverse of each element of G is unique.

Definition:

Let G be a group and $\emptyset \neq H \subseteq G$. If H is a group under the binary operation of G, then we call H a *subgroup* of G.

Note:

Let $(G,*)$ be a group. Then the two subgroups $(\{e\},*)$ and $(G,*)$ are called trivial subgroups or improper subgroups of $(G,*)$. All other subgroups of $(G,*)$ are called non-trivial subgroups or proper subgroups of $(G,*)$.

Theorem:

A non empty subset H of a group G is a subgroup of $(G,*)$ iff (i) $\forall a, b \in H, a * b \in H$ (ii) $\forall a \in H \Rightarrow a^{-1} \in H$.

Symmetric Group:

Let X be a non-empty set.A permutation of X is a one-one function from X to X.The set G of all permutations on a non-empty set X under the binary operation $*$ of the right composition of permutations is a group called permutation group.

If X $= \{1,2,3..n\}$, the permutation group is also called symmetric group denoted by S_n.The number of elements in $S_n$is $n!$.

Direct product of two groups:

Let (G, .) and (H, .) be groups. Define a binary operation $*$ on G×H by

$(g_1, h_1) \cdot (g_2, h_2) = (g_1 . g_2, h_1 * h_2)$ where $g_1, g_2 \in$ H.Then (G×H,·) is a group called direct product of G and H.

Definition:

If $(G, \circ)$ and $(H, *)$ are groups and $f: G \to H$, then f is called a *group homomorphism* if for all $a, b \in G$, $f(a \circ b) = f(a) * f(b)$.

Note:

Let $(G, \circ)$, $(H, *)$ be groups with respective identities e_G, e_H. If $f: G \rightarrow H$ is a homomorphism, then

a) $f(e_G) = e_H$. **b)** $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G$.

c) $f(a^n) = [f(a)]^n$ for all $a \in G$ and all $n \in \mathbf{Z}$.

d) $f(S)$ is a subgroup of H for each subgroup S of G.

Definition:

If $f: (G, \circ) \rightarrow (H, *)$ is a homomorphism, we call f an *isomorphism* if it is one-to-one and onto. In this case G, H are said to be *isomorphic groups*.

Definition:

A group G is called *cyclic* if there is an element $x \in G$ such that for each $a \in G$, $a = x^n$ for some $n \in \mathbf{Z}$.

Definition:

If H is a subgroup of G, then for each $a \in G$, the set $aH = \{ah | h \in H\}$ is called a *left coset* of H in G. The set $Ha = \{ha | h \in H\}$ is a *right coset* of H in G.

Theorem(Lagrange's Theorem):

If G is a finite group of order n with H a subgroup of order m,then m divides n.

TUTORIAL QUESTIONS

1. For each of the following sets, determine whether or not each is a group under the stated binary operation. If so, determine its identity and the inverse of each of its elements.

   (a) $\{10n \backslash n\ e\ Z\ \}$ under addition

   (b) $\{—\ 1, 1\}$ under addition

2. Define the binary operation o on Z by $x\ o\ y\ =\ x\ +\ y\ +\ l$. Verify that (Z, o) is an abelian group.
3. Let I be the set of integers.Show that the operation $*$ defined by $a * b = a + b + 1$ satisfies closure, associative and commutative law. Find the identity and the inverse.
4. Show that $(A, .)$ is a non abelian group where $A = R - \{0\} \rightarrow R$ and $(a, b).(c, d) = (ac, bc + d)$.
5. Is every Abelian group cyclic? Justify.

ASSIGNMENT QUESTIONS

1. For each of the following sets, determine whether or not the set is a group under the stated binary operation.

   (a) $\{—\ 1, 1\}$ under multiplication

(b) $\{—\ 1,0,1\}$ under addition

2. Why is the set Z *not* a group under subtraction?
3. Let $G\ =\ \{q\ e\ Q/q \neq -1\}$. Define the binary operation o on G by $x\ o\ y$ = $x + y + xy$. Prove that (G, o) is an abelian group
4. If $*$ is a binary operation on the set R of real numbers defined by $a * b = a + b + 2ab$. Find if $\{R,*\}$ is semigroup.Is it commutative.
5. Show that any group G is abelian iff $(ab)^2 = a^2b^2, \forall\ a, b \in G$.
6. Prove that every cyclic group is abelian.
7. Consider the group $G = \{1,2,3,4,5,6\}$ under modulo 7.

   (a) Find the multiplication table of G.

   (b) Find $2^{-1}, 3^{-1}, 6^{-1}$

   (c) Find the order and subgroup generated by 2 and 3.

   (d) Is G cyclic?
8. Verify that 2 is the generator of $(Z_4, +_4)$.
9. Let G be the group of non-zero complex numbers under multiplication, and let G' be the group of non-zero real numbers under multiplication. Check whether $f: G \to G'$ defined by $f(z) = |z|$ is homomorphism
10. Let G be the set of all 2×2 non-singular matrices over R and . be the matrix multiplication. Check whether $(G, .)$ is an abelian group.

UNIT WISE QUESTION BANK

1. Check whether Z,set of integers under the operation subtraction satisfies associative and commutative law.
2. Is $(N,*)$ a commutative monoid where $x * y = max\ \{x, y\}$
3. Show that $(A, .)$ is a non abelian group where $A = \{a \in Q: a \neq -1\}$ and for any $a, b \in A, a.b = a + b + ab$.
4. Find all subgroups of (a) z_{12} (b) S_3.
5. Let $A = [1\ 0\ 0\ 1\ ]$.

   a)Determine $A^2, A^3\ and\ A^4$.

   b) Verify that $\left\{A, A^2, A^3, A^4\right\}$ is an abelian group under ordinary matrix multiplication
6. Let $G = S_4$.

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

(a) For

Subgroup $H =< \alpha >$ **.**

(b) Determine the left cosets of H in G.

7. Let a be any element in a group G.Check whether the function $f: Z \to G$ defined by $f(n) = a^n$ is a homomorphism.
8. Give an example of a group **G** with subgroups ***H,*** *K* such that ***H*** **U** *K* is ***not*** a subgroup of **G.**
9. Prove that a group G is abelian if and only if for all $a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$.
10. Prove that if group **G** with subgroups ***H,*** *K* such that ***H*** **U** *K* is ***not*** a subgroup of **G.**
11. List the permutation group S_3
12. Prove that S_3 is an Abelian group.
13. Prove that every cyclic group is abelian.
14. Give an example of a semi group which is not monoid.
15. Find the subgroups of $(Z_6, +_6)$
16. Show that $(S_2, \circ)$ is monoid.
17. Let G be the set of all 2×2 non-singular matrices over R and $+$ be the matrix addition. Check whether $(G, +)$ is an abelian group
18. Let G be the group of real numbers under addition, and let G' be the group of positive real numbers under multiplication. Check whether $f: G \to G'$ defined by $f(a) = 2^a$ is homomorphism and isomorphic.
19. Verify that 3 is a generator of $(Z_4, +_4)$.
20. For $G = (Z_{24}, +)$, find the cosets determined by the subgroup $H =< [3] >$

Module 5

Graph Theory

What is a Graph?

- A graph G = (V, E) consists of a set of objects $V = \{v_1, v_2, v_3, \ldots\}$ called vertices and another set $E = \{e_1, e_2 \ldots\}$, whose elements are called edges.
- e_k is identified with an unordered pair (v_i, v_j) of vertices, called the edge, connecting v_i and v_j.

Self-loop, Parallel Edges and Simple graph

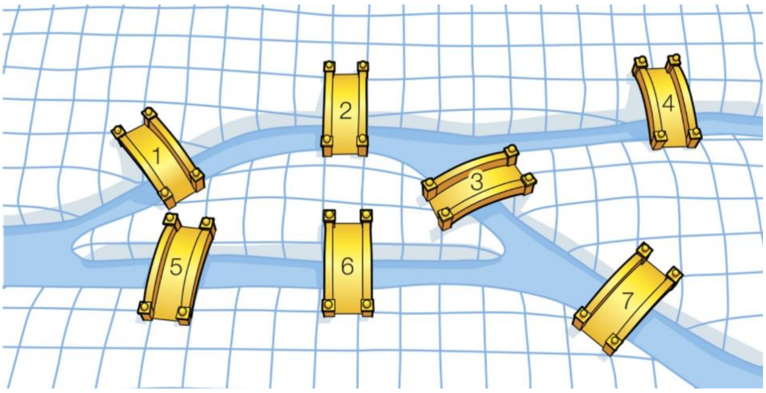
1. A self-loop (or simply loop) is an edge whose endpoints are the same vertex.
2. Multiple edges (or parallel edges) are edges having the same pair of endpoints. A graph with multiple edges is called as a multigraph.
3. A simple graph is a graph having no loops or multiple edges.

Konigsberg Bridge problem

The **KONIGSBERG BRIDGE** problem was a long-standing problem. It was solved by Leonhard Euler in 1736, by means of a graph.

PROBLEM: The two islands "A" and "D" formed by the Pregel River in Konigsberg were connected to each other and to the banks "B" and "C" with seven bridges as shown in the figure. The problem was to start at any of the four land areas of the city, A, B, C, D walk over each of the seven bridges exactly once, and return to the starting point.

Euler represented this situation by means of a graph as shown in the figure. The vertices represent the land areas, and the edges represent the bridges.



Euler proved that a solution for this problem does not exist.

Incidence and Degree

- When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with each other.

- Two non-parallel edges are said to be adjacent if they are incident on a common vertex.
- Two vertices are said to be adjacent if they are the end vertices of the same edge.

Directed Graph

- A directed graph or digraph G consists of a set of vertices V={$v_1, v_2, ...$}and a set of edges $E = \{e_1, e_2, ...\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).
- In the case of directed graph a vertex is represented by a point and edge by a line segment between v_i and v_j with an arrow directed from v_i to v_j.

First Theorem of Graph Theory

In any graph, the sum of degrees of all vertices is equal to twice the number of edges. (also called Handshaking Lemma).

Theorem: There are always an even number of odd vertices in a graph.

Regular Graph

- A simple graph in which all vertices are of equal degree is called a Regular Graph.
- If degree of each vertex of a simple graph is k, it is called a k-regular graph.

Null Graph

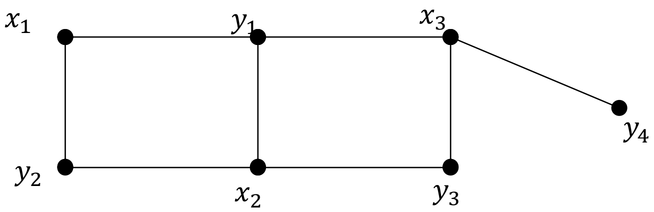
- If in a graph G = (V, E) if the edge set E is empty, is called a Null Graph.
- Every vertex in a null graph is an isolated vertex.

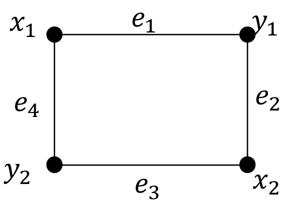
Complete Graph

- A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge.
- A complete graph of "n" vertices is denoted as k_n.

Bipartite Graph:

- Let $G = (V, E)$ be a graph, if the vertex set "V" can be partitioned into 2 non empty subsets X and Y (i.e. $V = X \, U \, Y$ & $X \cap Y = f$) in such a way that each edge of G has one end in "X" and other end in "Y". Then "G" is called a bipartite graph.
- The partition $V = X \, U \, Y$ is called a bipartition of G.





Complete Bipartite Graph

- A complete Bipartite graph is a simple bipartite graph "G" with bipartition $V = XUY$ in which every vertex in "X" is joined to every vertex of "Y".
- If "X" has "m" vertices and "Y" has "n" vertices such a graph is denoted by $K_{m,n}$and it has "mXn" edges, since each of the "m" vertices in the partition set "X" of $K_{m,n}$ is adjacent to each of the end vertices in the partition set "Y"

Weighted Graph

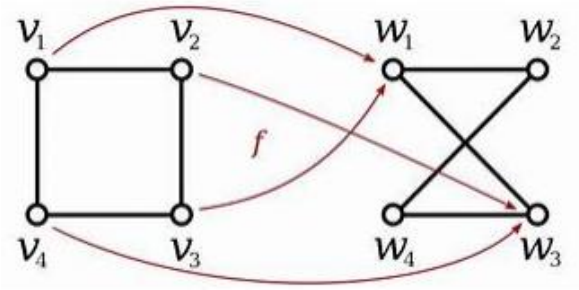
- In a graph G, with every edge e_i is associated with a real number (the distance in miles or time etc, say) or $w(e_i)$.Such a graph is called a weighted graph; $w(e_i)$being the weight of the edge e_i.

Subgraphs:

- A graph "H" is said to a subgraph of G if all the vertices and all the edges of "H" are in G and each edge of "H" has the same end vertices in "H" as in G. It is denoted as H ⊂ G, and is stated as "H" is a subgraph of G.

Isomorphism:

Two Graphs G and G' are set to be isomorphic (to each other), if there is a one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved. i.e. If an edge "e" in G is incident on the vertices v_1 and v_2 in G, then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' in G' that corresponds to v_1 and v_2 in G.



Walk:

- A <u>walk</u> is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with vertices preceding and following it.
- Vertex may appear more than once.
- It is possible for a walk to begin and end at the same vertex, such a walk is called a <u>closed walk</u>. A walk that is not closed (i.e., the terminal vertices are distinct) is called an <u>open walk</u>.
- A walk is also referred to as an <u>edge train or a chain</u>.
- Vertices with which a walk begins, and ends are called its <u>terminal vertices</u>

Path:

- An open walk in which no vertex appears more than once is called a path OR a simple path OR an elementary path.
- The number of edges in a path is called the length of the path.
- An edge which is not a self-loop is a path of length 1.
- A self-loop can be considered as a walk but not a path

Cycle:

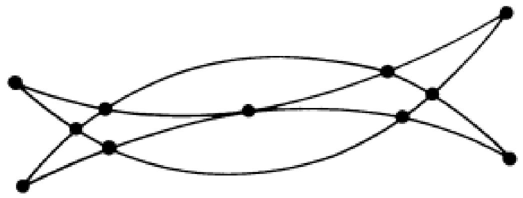
- A closed walk in which no vertex appear more than once is called a cycle. (Except the initial and final vertex).
- A cycle is also called a circuit, elementary cycle, circular path, and polygon.

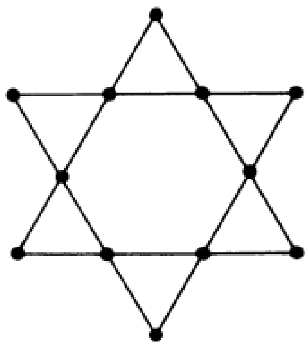
Connected graph

- A graph in which there is a path between any two vertices.

What is an Euler graph?

- If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph an Euler graph .



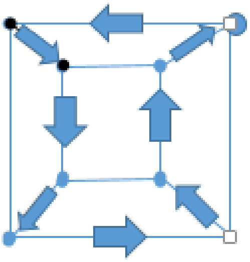


Theorem:

- A given connected Graph G is an Euler graph iff all vertices of G are of even degree.

Hamiltonian path and Hamiltonian Circuit

- A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except the starting vertex at which the walk terminates.



- If we remove any one edge from the Hamiltonian circuit, we are left with a path. This is called a Hamiltonian path.

Theorem:

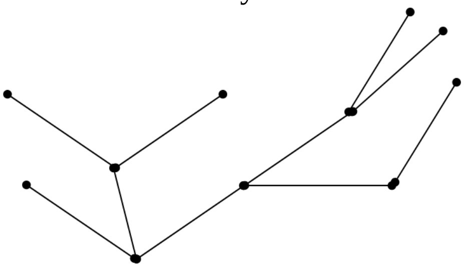
- In a complete graph with "n" vertices, there are $\frac{n-1}{2}$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

Dirac's Theorem:

- A sufficient condition for a simple graph G with $n \geq 3$ vertices have a Hamiltonian circuit is that the degree of every vertex in G be at least $\frac{n}{2}$, where n is the number of vertices in G.

Vertex Connectivity

- The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the graph disconnected.
- Vertex connectivity of a tree is 1.

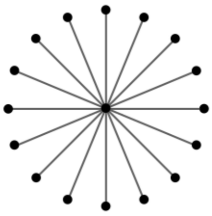
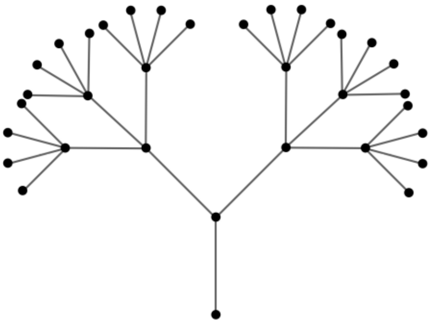
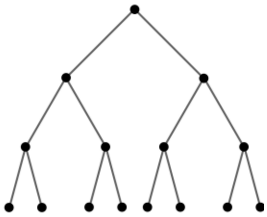


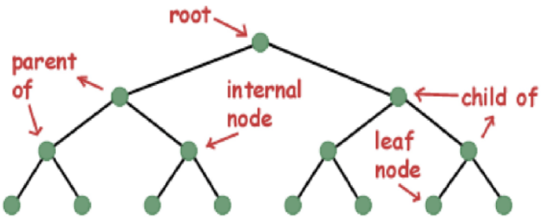
Edge Connectivity

- The edge connectivity of a connected graph can be defined as the minimum number of edges whose removal (deletion) reduces the rank of the graph by 1.
- The edge connectivity of a tree is 1

Tree

Tree is a simple connected acyclic graph.



Properties of Trees:

- There is one and only one path between every pair of vertices in a tree T.
- If in a graph G there is one and only path between every pair of vertices, G in a tree.
- A tree with "n" vertices has $(n-1)$ edges.
- Any connected graph with "n" vertices and $n-1$ edges is a tree.
- A graph G with "n" vertices, $(n-1)$ edges, and no circuit is connected.
- In any tree (with two or more vertices) there are at least two pendant vertices.

Spanning Tree

- A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

Fleury's algorithm

- Fleury's Algorithm is used to display the Euler circuit in any given connected graph in which each vertex has even degree.

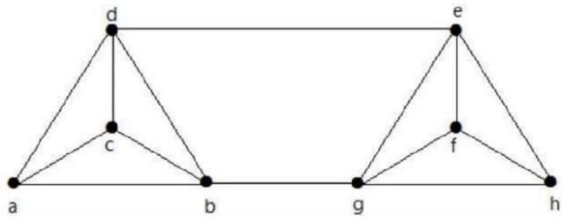
- In this algorithm, starting from one edge, it tries to move other adjacent vertices by removing the previous vertices. Using this trick, the graph becomes simpler in each step to find the Euler path or circuit.

Algorithm:

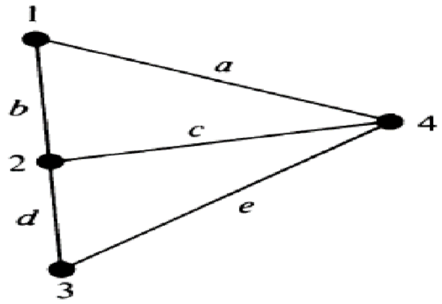
- **Step I** : Start at any vertex. Go along any edge from this vertex to another vertex. Remove this edge from the graph.
- **Step II** : You are now on a vertex on the revised graph. Choose any edge from this vertex, but not a cut edge, unless you have no other option. Go along your chosen edge. Remove this edge from the graph.
- **Step III** : At step II until you have used all the edges and get back to the vertex at which you started.

TUTORIAL QUESTIONS

1. Let G be k-regular graph where k is an odd number. Prove that the number of edges in G is a multiple of k.
2. What is the smallest number n such that the complete graph Kn has at least 500 edges?
3. Can a graph be both Hamiltonian and Eulerian? Justify.
4. Calculate the vertex connectivity, edge connectivity and minimum vertex degree of the following graph.

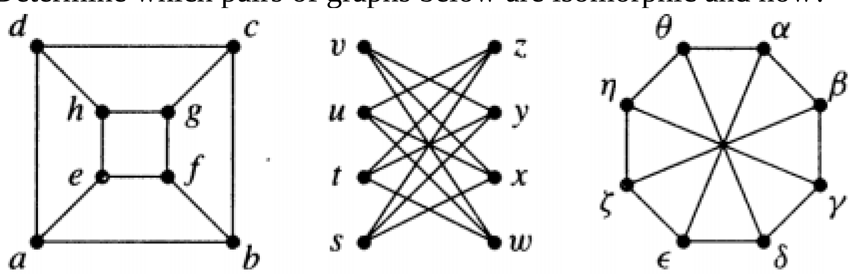


5. Sketch all spanning trees of the graph

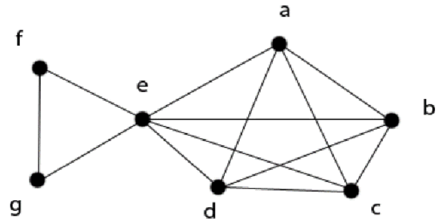


ASSIGNMENT QUESTIONS

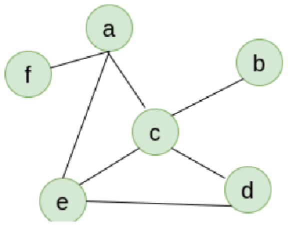
1. Draw a graph on 6 vertices with degree sequence (3, 3, 5, 5, 5, 5), does there exist a simple graph with these degrees? If yes, draw the graph, otherwise, give reason. How are your answers changed if the degree sequence is (2, 3, 3, 4, 5, 5)?
2. Determine which pairs of graphs below are isomorphic and how?



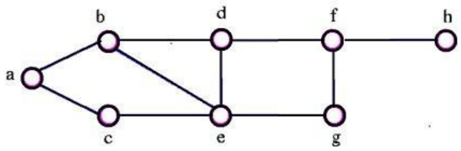
3. Draw a simple graph with 8 vertices, 4 components and maximum number of edges.
4. Let G be a simple regular graph with n vertices and 24 edges. Find all possible values of n and give examples of G in each case.
5. Check whether the given graph is Euler graph and if yes, give the Euler walk. Justify.



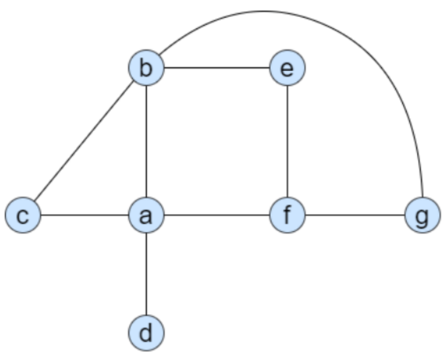
6. Does the graph in above question has Hamiltonian path? If yes, give a Hamiltonian path.
7. Draw all non-isomorphic trees with six vertices.
8. Find the Eccentricity, Diameter, Radius and Centre of the graph shown below:



9. Find all spanning trees of the graph



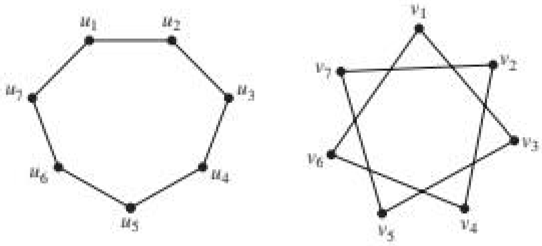
10. Use Fleury's Algorithm to find Euler path or Euler Circuit in the following graph



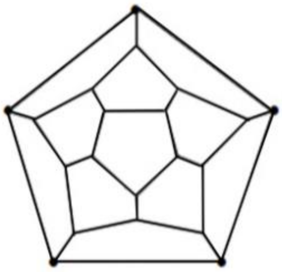
QUESTION BANK

1. Can a simple graph exist with 15 vertices, with each of degree 5? Justify your answer.
2. Let G be a graph with n vertices, t of which have degree k and the others have degree k+1. Prove that t = (k+1)n – 2e.
3. Draw a graph with four edges, four vertices having degree 1, 2, 3, 4. If not explain why no such graph exists.
4. Draw all simple graph of one, two, three, and four vertices.
5. Draw a connected graph that becomes disconnected when any edge is removed from it.

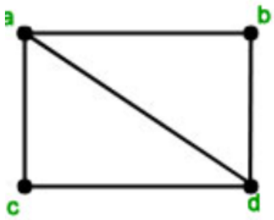
6. What is the maximum number of edges in a simple graph with 'n' vertices?
7. Define isolated vertex, pendent vertex, even vertex and odd vertex. Draw a graph that contains all the above.
8. Determine whether the given pairs of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



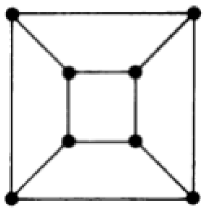
9. Is Hamiltonian circuit always implying Hamiltonian path? Is the converse true? Justify your answer.
10. Is the converse of Dirac's theorem true? Justify your answer.
11. Is Petersen graph Hamiltonian? Justify your answer.
12. Does the following graph have a Hamiltonian Circuit?



13. Check whether the graph is Eulerian graph. Justify your answer.



14. For which values of m, n is the complete graph $K_{m,n}$ an Euler graph ? Justify your answer.
15. Draw all non-isomorphic trees with seven vertices.
16. For what value of 'n' do the complete graph Kn have an Euler circuit ? Give the reason.
17. Find the smallest integer n such that the complete graph has at least 500 edges.
18. State Dirac's theorem and check the applicability of the following graph.



19. Find the number of edge-disjoint Hamiltonian circuits in the complete graph with 9 vertices. Draw all of its edge-disjoint Hamiltonian circuits.
20. Explain Fleury's Algorithm.